Using the ι (iota) operator:

It is important to remember that the iota operator is just a handy notational shortcut that allows us to write some sentences more quickly and arguably, more perspicuously.

Grammatically, ιxFx is a term like a name (Joel) or a function applied another term (the father of the father of Joel). But it isn't some specific term that can be replaced in a one-to-one fashion in a larger sentence. It is essentially incomplete.

The definition is RE on page 43 of MacFarlane.

So for example, for a one place predicate:

TιxFx is really just a shortcut for ∃x(Fx ∧ ∀y(Fy ⊃ y = x) ∧ Tx) [[The farmer is tall]]

A two place relation:

RιxFxa is really just a shortcut for ∃x(Fx ∧ ∀y(Fy ⊃ y = x) ∧ Rxa) [[The farmer respects Alice]]

Remember that an identity sentence like a=b is really just itself a notational variation on a two place relation. So for 'ιxFx = a' just imagine it said IιxFxa, use the translation above, and then instead of Ixa write x=a. So 'The president is Joe Biden' would be ιxPx = j which is ∃x(Px ∧ ∀y(Py ⊃ y = x) ∧ x = j). Now you might think this should obviously be translated as 'Pj' where 'Px' just means 'x is the president'. But logically, Pj does not entail that there is only one P. So if you think that 'The president is Joe Biden' really does entail that there is just one president, then Pj is logically lacking something important.

Now neither the formula that the thing uniquely satisfies (ϕx) nor the formula you are saying that thing must also satisfy (ψx) have to be atomic. Either could itself be something complicated. An example of a conditional $Fx \supset Hx$ is on page 44. There we have:

$\exists x(Gx \wedge \forall y(Gy \supset y{=}x) \wedge (Fx \supset Hx))$

represented by [ιxGx]($F\ \iota xGx \supset H\ \iota xGx$)

That case introduces an important fact about scope. The brackets outside of the conditional indicate that the scope of the iota operator covers the whole conditional. That means a single existential quantifier will bind BOTH of the ιx s in the conditional.

Here is more complicated example.

The farmer loves the president.

LιxFx ιyPy

Start by saying there is a unique F:

$\exists x(Fx \wedge \forall y(Fy \supset y = x) \wedge$

Now we need to say that this thing 'x' loves the president.

$\exists x(Fx \wedge \forall y(Fy \supset y = x) \wedge \exists y(Py \wedge \forall z(Pz \supset z = y) \wedge Lxz))$

But if I wanted to say that the farmer loves himself that would be much easier because we want a single quanitifer to bind both instances of 'the F':

$[\iota xFx]\ L\iota xFx\ \iota xFx$

$\exists x(Fx \wedge \forall y(Fy \supset y = x) \wedge Lxx)$

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When thinking about arguments involving iota sentences, you should think that they are really the longer sentences and then ask if they are valid. To do this, we just add a rule that allows us to move from the iota sentence to its Russellian equivalent and vice versa.

So this is valid:

The president is a man
Every man is tall
Therefore, the president is tall

Here is the logical form using iota sentences:

$M\iota xPx$
$\forall x(Mx \supset Tx)$
$T\iota xPx$

So how can we do this proof? What you cannot do is plug in 'ιxPx' to line 2 and then do Modus Ponens. That would be logically invalid. But instead what you do is just turn the first sentence into its ordinary first order representation, then prove the thing which is the ordinary representation of the conclusion. MacFarlane calls this transformation 'RE'

Here is the proof:

1) $M\iota xPx$ premise
2) $\forall x(Mx \supset Tx)$ premise
3) $\exists x(Px \wedge \forall y(Py \supset y = x) \wedge Mx)$ by RE from 1
4) new scope - box a $Pa \wedge \forall y(Py \supset y = a) \wedge Ma$ hyp - set up for EElim

5) Ma $\supset$ Ta Universal Elim 2
6) Ta modus ponens 4,5
7) Pa $\wedge$ $\forall$y(Py $\supset$ y = a) $\wedge$ Ta $\wedge$ intros 4,6
8) $\exists$x(Px $\wedge$ $\forall$y(Py $\supset$ y = x) $\wedge$ Tx) Ex Intro
9) end scope $\exists$x(Px $\wedge$ $\forall$y(Py $\supset$ y = x) $\wedge$ Tx) EElim
10) TιxPx RE from 9